Exercise 24

Prove that $\lim_{x\to 0} x^2 \cos(1/x^2) = 0$.

Solution

Solution Using a Substitution

Rewrite the left side.

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = \lim_{x\to 0} \frac{1}{\frac{1}{x^2}} \cos\left(\frac{1}{x^2}\right)$$

$$= \lim_{x\to 0} \frac{\cos\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

Make the substitution, $u = 1/x^2$. Note that as $x \to 0$, $u \to \infty$.

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = \lim_{u\to\infty} \frac{\cos u}{u}$$

$$= 0$$

This limit is zero because the denominator gets bigger and bigger as $u \to \infty$, whereas the numerator remains bounded between -1 and 1.

Solution Using the Squeeze Theorem

Recognize that $\cos(1/x^2)$ is bounded by -1 and $+1$, so we can consider the inequality,

$$-x^2 \le x^2 \cos\left(\frac{1}{x^2}\right) \le x^2.$$

Take the limit of all sides as $x \to 0$.

$$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le \lim_{x\to 0} x^2$$

Evaluate the limits on the left and right sides.

$$-(0)^2 \le \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le (0)^2$$

This means

$$0 \le \lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) \le 0.$$

Therefore, by the Squeeze Theorem,

$$\lim_{x\to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0.$$